B.TECH. SEM-I · Test-2.

Engineering Mathematics - I.

1) a) n^{th} derivative of $y = \frac{x^2}{(x+2)(2x+3)}$.

$$y = \frac{\frac{1}{2}(2x^2+7x+6) - \left(\frac{7x+6}{2}\right)}{2x^2+7x+6}$$

$$y = \frac{1}{2} - \frac{1}{2}\,\frac{7x+6}{(x+2)(2x+3)}.$$

Use partial fracⁿ for Second term.

$$y = \frac{1}{2} - \frac{1}{2}\left[\frac{8}{x+2} - \frac{9}{2x+3}\right].$$ —— (3 Marks)

$$y_n = \cancel{0 - \tfrac{1}{4}}\; 0 - 4\,\frac{(-1)^n . n!}{(x+2)^{n+1}} + \frac{9}{2}\,\frac{(-1)^n . n!\, 2^n}{(2x+3)^{n+1}}$$

—— (3 marks)

b) $y = a\cos(\log x) + b\sin(\log x)$

$y_1 = -a\sin(\log x)\times\left(\frac{1}{x}\right) + b\cos(\log x)\cdot\left(\frac{1}{x}\right)$

$xy_1 = -a\sin(\log x) + b\cos(\log x).$

$xy_2 + y_1 = -a\cos(\log x)\cdot\left(\frac{1}{x}\right) - b\sin(\log x)\cdot\left(\frac{1}{x}\right).$

$x^2y_2 + xy_1 = -y.$ —— (3 marks)

$x^2y_2 + xy_1 + y = 0$

diff. n times w.r.t. x.

$(x^2y_2)_n + (xy_1)_n + y_n = 0.$

$\left[x^2y_{n+2} + 2n\cdot x y_{n+1} + (n^2-n)y_n\right] + (xy_{n+1} + ny_n) + y_n = 0.$

$x^2y_{n+2} + (2n+1)\,x y_{n+1} + (n^2+1)\,y_n = 0.$

—— (3 Marks)

c) Lagrange's Mean Value thm

$f(x) = 2x^3 - 7x + 10 \quad ; 2 \le x \le 5$

being a polynomial function.

Continuous in $[2,5]$ & differentiable on $(2,5)$

$\therefore$ All conditions of LMVT are satisfied.

$\therefore \exists$ at least one pt $c \in (2,5)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b-a} \quad \text{——— (2 marks)}$$

$f(x) = 2x^3 - 7x + 10$

$f'(x) = 6x^2 - 7$

$f(5) = 225$

$f(2) = 16 - 14 + 10 = 2 + 10 = 12$

$$f'(c) = \frac{f(5) - f(2)}{5-2}$$

$$6c^2 - 7 = \frac{225 - 12}{3}$$

$$= \frac{213}{3}$$

$6c^2 - 7 = 71$

$6c^2 = 71 + 7$

$c^2 = \frac{78}{6}$

$c^2 = 13$

$c = \pm\sqrt{13}$ ——— (2 marks)

$\therefore c = \sqrt{13} \in (2,5)$ Hence LMVT satisfied.

OK

2a) $y = \cos x \cdot \cos 2x \cdot \cos 3x$

$= \frac{1}{2} \cos 2x \left[2 \cos x \cdot \cos 3x\right]$

$= \frac{1}{2} \cos 2x \left[\cos 4x + \cos 2x\right]$

$= \frac{1}{2} \left[\cos 2x \cdot \cos 4x\right] \frac{1}{2} \cos^2 2x$

$= \frac{1}{4} \left[2 \cos 2x \cdot \cos 4x\right] + \frac{1}{2} \cos^2 2x$

$= \frac{1}{4} \left[\cos 6x + \cos 2x\right] + \frac{1}{2} \left(\frac{1 + \cos 4}{2}\right.$

$= \frac{1}{4} \left[\cos 6x + \cos 4x + \cos 2x + 1\right]$ —— (03)

Now, differentiate n times

$y_n = \frac{1}{4} \left[6^n \cos\left(6x + \frac{n\pi}{2}\right) + 4^n \cos\left(4x + \frac{n\pi}{2}\right) + 2^n \cos\left(2x + \frac{n\pi}{2}\right)\right]$ —— (03)

2b) $y = \cos (m \log x)$

Diff w.r.t 'x'

$y_1 = -\frac{m}{x} \sin(m \log x)$

$x y_1 = -m \sin(m. \log x)$

Diff w.r.t x.

$x y_2 + y_1 = \frac{m^2}{x} \cos(m. \log x)$

$x^2 y_2 + x y_1 = -m^2 \cos(m. \log x)$

$= -m^2 y$

$x^2 y_2 + x y_1 + m^2 y = 0$ —— (03)

Now differentiating n times w.

$(x^2y_2)_n + (xy_1)_n + m^2 y_n = 0$ — ①

Consider $(x^2y_2)_n = (u \cdot v)_n$

$u = y_2$ $v = x^2$

$u_n = y_{n+2}$ $v_1 = 2x$

$u_{n-1} = y_{n+1}$ $v_2 = 2$

Using leibnitz's theorem,

$y_n = (uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2$

$(x^2y_2)_n = y_{n+2}(x^2) + n y_{n+1}(2x) + \frac{n(n-1)}{2}(2) y_n$

consider $(xy_1)_n = (uv)_n$

$(xy_1)_n = y_{n+1} x + n y_n 1$

$= x y_{n+1} + n y_n$

~~Consider~~, from ①

$[x^2 y_{n+2} + 2n x y_{n+1} + (n^2 - n) y_n]$

$+ [x y_{n+1} + n y_n] + m^2 y_n = 0$

$x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2 + m^2) y_n = 0$

2c) Here $f(x) = x^2(1-x^2)^2$, $0 \le x \le 1$

1) given function is continuous in $[0,1]$ being polynomial function in x.

2) $f'(x) = 2x(1-x^2)^2 - 2(1-x^2)(-2x) x^2$

which does not become infinite or indeterminate at any pt of the interval $(0,1)$

... differentiable

c) $f(0) = 0 = f(1)$

Hence the condition of Rolle's theorem are satisfied. Therefore There exist atleast one pt $c \in (0,1)$ such that $f'(c) = 0$ —— (02 Marks)

$$f(x) = x^2[1 - 2x^2 + x^4]$$

$$= x^2 - 2x^4 + x^6$$

$$f'(x) = 2x - 8x^3 + 6x^5$$

$$f'(c) = 2c - 8c^3 + 6c^5 = 0$$

$$c - 4c^3 + 3c^5 = 0$$

$$c - 3c^3 - c^3 + 3c^5 = 0$$

$$c(1 - 3c^2) - c^3(1 - 3c^2) = 0$$

$$(1 - 3c^2)(c - c^3) = 0$$

$$(1 - 3c^2)\, c(1 - c^2) = 0$$

$$c^2 = \frac{1}{3}, \cancel{1}, \quad c^2 = 1, \quad c = 0$$

$$c = \pm\frac{1}{\sqrt{3}}, \quad c = \pm 1, \quad c = 0$$

Out of these $\frac{1}{\sqrt{3}} \in (0,1)$ —— (02 Marks)

Q3 a) Range of Convergence.

$$\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2x^2+\left(\frac{4}{5}\right)^3x^3+\cdots$$

$$U_n = \left(\frac{n}{n+1}\right)^{n-1}x^{n-1}, \quad U_{n+1}=\left(\frac{n+1}{n+2}\right)^{n}x^{n}.$$

$$\lim_{n\to\infty}\frac{U_n}{U_{n+1}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n-1}x^{n-1}\times\left(\frac{n+2}{n+1}\right)^{n}\frac{1}{x^n}$$

$$= \lim_{n\to\infty}$$

$$= \frac{1}{x}$$ ——— (2 Marks)

By Ratio Test.

If $\frac{1}{x}>1$, Series is convergent
i.e. $x<1$.

If $1/x<1$, Series is divergent
i.e. $x>1$.

(2 Marks)

& If $x=1$, Ratio test fails.

For $x=1$, $U_n=\left(\frac{n}{n+1}\right)^{n-1}$

$$\lim_{n\to\infty}U_n = \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^{n-1} = e \neq 0$$

$\therefore$ Series is divergent if $x=1$. ——— (2 Marks)

$\therefore$ For $x<1$, Series is cgt.
for $x\geq 1$, series is dgt.

$$y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right) = \cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right).$$

Put $x = \tan\theta$

$$y = \cos^{-1}\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right) = \cos^{-1}\left(-\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right)\right)$$

$$= \cos^{-1}(-\cos 2\theta) = \cos^{-1}(\cos(\pi - 2\theta))$$

$$= \pi - 2\theta = \pi - 2\tan^{-1}x$$ (3 Marks)

$$y = \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)$$ (1 Mark)

---

3 c) Expand $\tan^{-1}x$ in Powers of $(x-1)$.

$f(x) = \tan^{-1}x, \quad a = 1.$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

$$= f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \cdots$$ 1 Mark

$f(x) = \tan^{-1}x, \qquad f(1) = \tan^{-1}(1) = \pi/4.$

$f'(x) = \dfrac{1}{1+x^2} \qquad f'(1) = 1/2$

$f''(x) = \dfrac{-2x}{(1+x^2)^2} \qquad f''(1) = -1/2.$

$f'''(x) = \qquad f'''(1) = 1/2.$

(2 Marks)

$$f(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 + \cdots$$

(1 Mark)

Q4 (a) $\sec^{-1}\left(\frac{1}{1-2x^2}\right) = \cos^{-1}(1-2x^2)$.

Put $x = \sin\theta$,

$1-2x^2 = 1-2\sin^2\theta = \cos(2\theta)$

$\therefore \cos^{-1}(1-2x^2) = \cos^{-1}(\cos(2\theta)) \cancel{= 2\theta = 2s}$

$= \cos^{-1}(\cos(2n\pi+2\theta))$

$= 2n\pi + 2\theta = 2n\pi + 2\cdot\sin^{-1}x$ (4 Marks)

$= 2n\pi + 2\left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} - \dots\right)$

(2 Marks)

---

4 b) $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} \dots$

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$

$u_n = \frac{1}{n\sqrt{n}}$.

$n\sqrt{n} < (n+1)\sqrt{n+1} \quad \forall n$

$\frac{1}{n\sqrt{n}} > \frac{1}{(n+1)\sqrt{n+1}} \quad \forall n$.

$u_n > u_{n+1} \quad \forall n$. —— (2 Marks)

& $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0$.

By Leibnitz's test, —— (2 marks)

$\therefore$ This Alternating series is convergent.

---

4 c) $\sum_{n=1}^{\infty} \sqrt{n^3+1} - \sqrt{n^3-1}$

$u_n = \frac{\sqrt{n^3+1} - \sqrt{n^3-1}}{\sqrt{n^3+1} + \sqrt{n^3-1}} \times \sqrt{n^3+1} + \sqrt{n^3-1}$

$u_n = \frac{2}{\sqrt{n^3+1} + \sqrt{n^3-1}}$, choose $v_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{2}{\sqrt{n^3+1}+\sqrt{n^3-1}} \times \sqrt{n^3} = 1 \quad \text{— 2 marks}$$

Which is finite & non-zero.

$\therefore$ $\sum U_n$ is cgt by Comparison test.

Since $\sum V_n$ is cgt by P-test.

$P = 3/2 > 1$. (2 marks)