Paper Code - U127-101(ESE)

May 18 solution.

Engg. mathematics II

Q.1 a) Let

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$, be the eqn of sphere

This passes through (1,0,0)

$\Rightarrow 1+2u+d=0$

$\therefore u = \frac{d+1}{-2}$

Similarly, $v = \frac{1+d}{-2}$, $w = \frac{1+d}{-2}$ (2)

We know that $r^2 = u^2+v^2+w^2-d$

$r^2 = \frac{3(d+1)^2}{4} - d = \frac{3(d^2+2d+1)-4d}{4}$

$\therefore r^2 = \frac{3d^2+2d+3}{4} = f(d)$, say

For least possible radius $f'(d)=0$

$\therefore 6d+2=0$

$\therefore d = -1/3$

$\therefore 2u=2v=2w=-2/3$ (3)

Hence the required eqn of sphere is -

$x^2+y^2+z^2-2/3(x+y+z)-1/3=0$ (01)

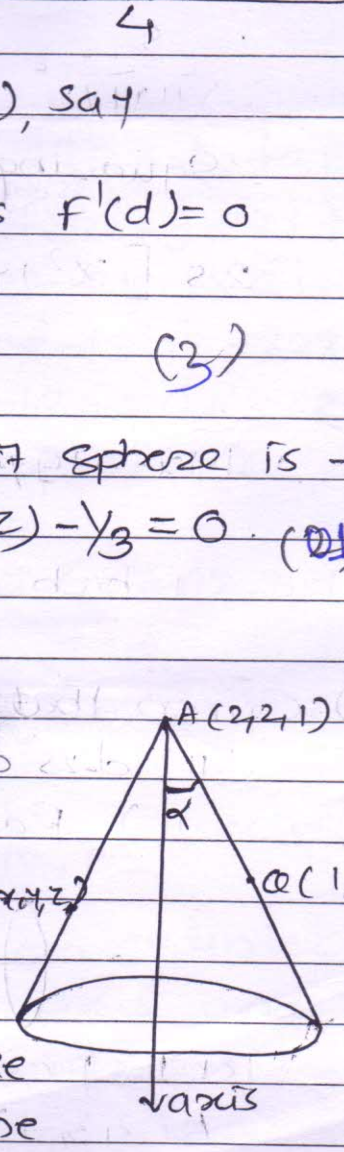
b) Given that

i) vertex at A(2,2,1)

ii) Axis is parallel to the line

$\frac{x+1}{2} = \frac{y-1}{-2} = \frac{z-2}{3}$

$\therefore$ dr's of axis are, 2, -3, 3 (2)



Let α be the semivertical angle

$\therefore \alpha$ is the angle betn axis of cone and AQ.

$$\therefore \cos\alpha = \frac{-2-2+9}{\sqrt{1+1+9}\,\sqrt{4+4+9}} = \frac{5}{\sqrt{11}\sqrt{17}} = \frac{5}{\sqrt{187}}$$ — 2

using the standard procedure

Step 1. Let $p(x,y,z)$ be any point on generator

Step 2. Given that $A(2,2,1)$

$\therefore$ d.r.s of Ap are $x-2, y-2, z-1$

Step 3. d.r.s of axis are $2,-2,3$.

Step 4.

use formula for

$$\cos\alpha = \frac{2(x-2)-2(y-2)+3(z-1)}{\sqrt{4+4+9}\,\sqrt{(x-2)^2+(y-2)^2+(z-1)^2}}$$ (2) — 2

$$\therefore \frac{5}{\sqrt{11}\sqrt{17}} = \frac{2x-4-2y+4+3z-3}{\sqrt{17}\sqrt{(x-2)^2+(y-2)^2+(z-1)^2}}$$

Squaring we get,

$$25\left[x^2+y^2+z^2-4x-4y-2z+9\right] = 11\left[2x-2y+3z-3\right]^2$$
$$= 11\left[4x^2+4y^2+9z^2+9-8xy+12xz-12x-12yz-12y-18z\right]$$

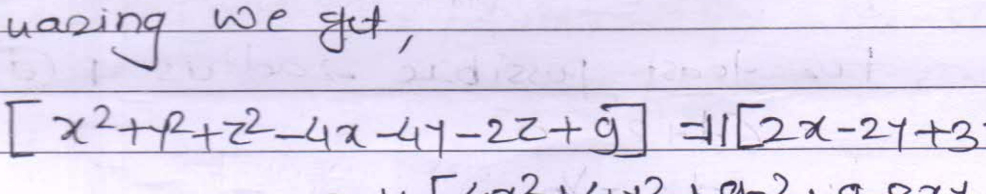
$$\therefore 19x^2+19y^2+74z^2-88xy+132xy-132yz-32x-32y-148z-126=0.$$

which is the required eqn of RCC. (2) — 2

c) Given that i) one point on axis $=(2,3,-1)$

ii) d.r.s of axis $3,1,1$

iii) Radius of RCC is 5



Step 1. $P(x,y,z)$ be any point on cylinder & $A(2,3,-1)$ be one point on axis and M be the $\perp$er from p on axis

Step 2. $AP = \sqrt{(x-2)^2+(y-3)^2+(z+1)^2}$

$PM = 5$

d.r.s of axis are $3, 1, 1$

$\therefore$ direction cosines of axis are $\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}$

$\therefore$ AM = projection of AP on axis AB.

$$= \frac{3}{\sqrt{11}}(x-2) + \frac{1}{\sqrt{11}}(y-3) + \frac{1}{\sqrt{11}}(z+1) \quad (2)$$

$$AM = \frac{3x+y+z-8}{\sqrt{11}} \quad \text{———} \quad 2$$

Step 3. From ΔAPM, $AP^2 = AM^2 + MP^2$

$$(x-2)^2 + (y-3)^2 + (z+1)^2 = \frac{(3x+y+z-8)^2}{11} + 25$$

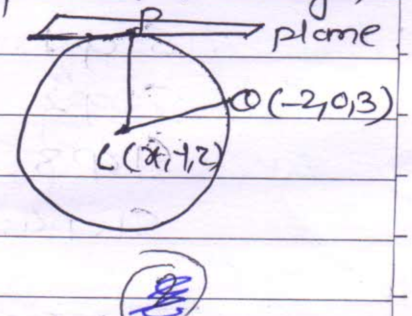
$$2x^2 + 10y^2 + 10z^2 - 6xy - 6xz - 2yz + 4x - 50y$$
$$+ 38z - \cancel{222}\,185 = 0 \quad 2$$

Q.2 a)

Given plane is tangent to the required sphere at point $P(-3,4,2)$

$\therefore$ CP is normal to the plane $2x-2y-z+16=0$

$\therefore$ d.r.s of CP are $2, -2, -1$, which passes through $P(-3,4,2)$



$\therefore$ Equation of CP is

$$\frac{x+3}{2} = \frac{y-4}{-2} = \frac{z-2}{-1} = k \text{ say}$$

$$x = 2k-3, \quad y = -2k+4, \quad z = -k+2$$

$\therefore$ The coordinates of centre C are

$$(2k-3, \ -2k+4, \ -k+2) \quad \text{———} \quad 2$$

$\therefore CP^2 = 4k^2 + 4k^2 + k^2 = 9k^2$

$CP^2 = CO^2$

$\therefore 9k^2 = (2k-3+2)^2 + (-2k+4-0)^2 + (-k+2-3)^2$ (2)

$\therefore k = 1$

centre is at $(-1, 2, 1)$ & radius $= \sqrt{9k^2} = 3$

$\therefore$ required eqn of sphere is -

$(x+1)^2 + (y-2)^2 + (z-1)^2 = 9$ (2)

$\therefore x^2 + y^2 + z^2 + 2x - 4y - 2z - 3 = 0$

b)

Given that

i) vertex is at origin $O(0,0,0)$

ii) Axis makes constant angles with coordinate axes

$\therefore$ eqn of axis is $\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{1}$

i.e. $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$

$\therefore$ d.r.s of axis are $1, 1, 1$

iii) d.r.s of generator of cone are $1, -2, 2$

$\therefore \cos\theta = \frac{1-2+2}{\sqrt{3}\sqrt{1+4+4}} = \frac{1}{3\sqrt{3}}$ (2)

where θ is semivertical angle.

Step 1. $P(x,y,z)$ be any point on generator

Step 2. Given $O(0,0,0)$ $\therefore$ d.r.s of $\overline{OP}$ are x, y, z

Step 3. d.r.s of axis are $1, 1, 1$

Step 4. Using formula for $\cos\theta$ (2)

$\frac{1}{3\sqrt{3}} = \cos\theta = \frac{x+y+z}{\sqrt{1^2+1^2+1^2}\sqrt{x^2+y^2+z^2}}$

on squaring we get

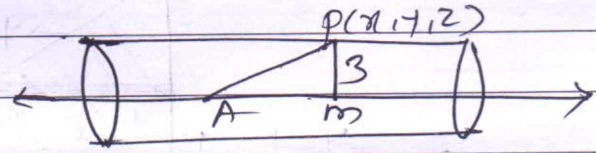
$4(x^2+y^2+z^2) + 9(xy+xz+yz) = 0$ (2)

which is required eqn of RCC

Q.2 C)

Given that the eqn of axis of RCC is

$$\frac{x-1}{1} = \frac{y-(-2)}{2} = \frac{z-0}{2}$$



∴ we have

① one point on axis = $A(1,-2,0)$

② d.r's of axis are $1, 2, 2$

③ Radius of cylinder $= 3$

Step 1) $P(x,y,z)$ be any point on RCC

$\therefore AP^2 = (x-1)^2 + (y+2)^2 + (z-0)^2$

$PM = 3$, dcs of axis are $1/3, 2/3, 2/3$

Step 2)

$AM =$ projection of AP on axis

$= \frac{1}{3}(x-1) + \frac{2}{3}(y+2) + \frac{2}{3}(z-0)$

$= \frac{1}{3}(x+2y+2z+3)$ 2

Step 3) From ΔAPM, $AP^2 = AM^2 + MP^2$

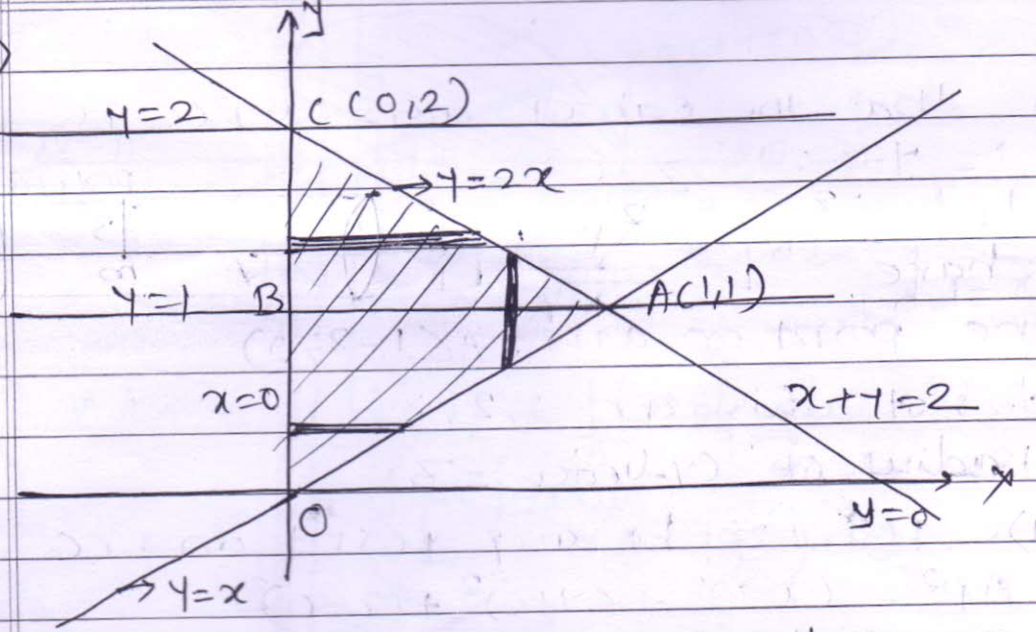
$$(x-1)^2 + (y+2)^2 + z^2 = \frac{1}{9}(x+2y+2z+3)^2 + 9$$

$$\therefore 8x^2 + 5y^2 + 5z^2 - 4xy + 4xz + 8yz - 24x + 24y - 12z - 90 = 0$$

which is required eqn of RCC. 2

6/12

Q.3 a)



2

The limits for I_1 and I_2 are

$x=0,\ x=y$ $\qquad$ $x=0,\ x=2-y$

$y=0 \quad y=1$ $\qquad$ $x+y=2$

$x=1,\ y=2$

$$\therefore\ I = \int_{x=0}^{1}\int_{y=x}^{2-x} (x^2+y^2)\,dx\,dy = \int_0^1 \left(x^2y + y^3/3\right)_x^{2-x} dx \quad 2$$

$$= \int_0^1 \left[\left\{x^2(2-x) + \frac{(2-x)^3}{3}\right\} - \left\{x^3 + x^3/3\right\}\right] dx$$

$$= \int_0^1 \left(-8/3\, x^3 + 4x^2 - 4x + 8/3\right) dx$$

$$I = 4/3 \quad 2$$

b) $$I = \int\int_{y^2}^{1} x\int_{z=0}^{1-x} dz\,dx\,dy = \int_0^1\int_{y^2}^{1} x\,(z)_0^{1-x}\,dx\,dy$$

$$= \int_0^1\int_{x=y^2}^{1} x(1-x)\,dx\,dy = \int_0^1 \left(x^2/2 - x^3/3\right)_{y^2}^{1}$$

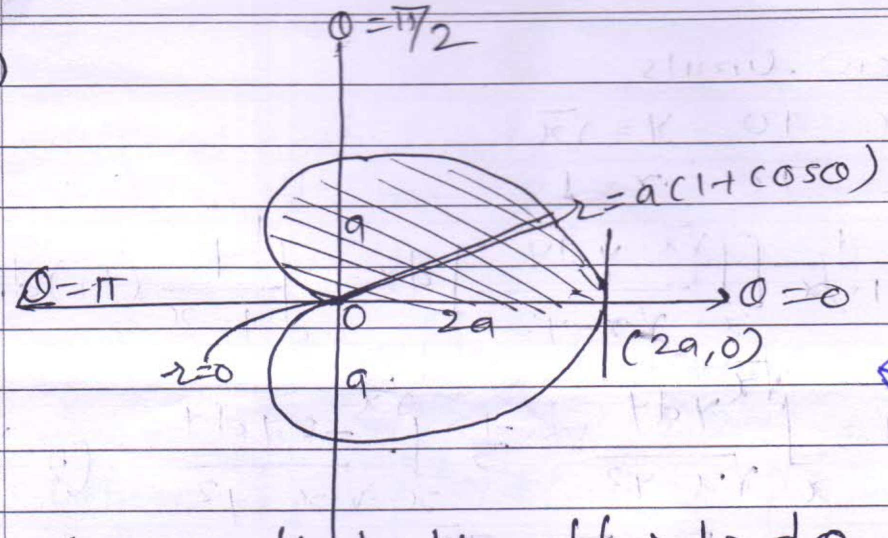
$$= \int_0^1\left[(1/2 - 1/3) - \left(y^4/2 - y^6/3\right)\right] dy = \left[1/6\, y - y^5/10 + \frac{y^7}{21}\right]_0^1$$

$$I = 4/35 \quad 1$$

6

1/10

c)



1

$$\text{Area} = \iint_A dx\,dy = \iint_A r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\left(\int_{r=0}^{a(1+\cos\theta)} r\,dr\right)d\theta$$ 1

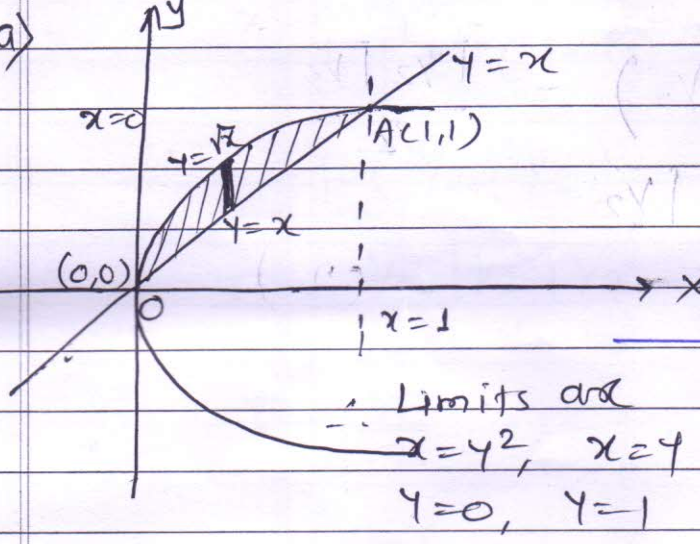
$$= \frac{1}{2}\int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left[1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right]d\theta$$

$$= \frac{a^2}{2}\left[\theta+2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$\text{Area} = \frac{a^2}{2}\left[\frac{3\pi}{2}+0\right] = \frac{3\pi a^2}{4}$$ 2

Q.4a)



$$I = \int_0^1 y\left[\int_{x=y^2}^{x=y}\frac{dx}{(1-x)\sqrt{x-y^2}}\right]dy$$ 1

2

∴ Limits are
$x=y^2$, $x=y$
$y=0$, $y=1$

$y^2=x$, put $y=x$ to get coordinates of point A

∴ $x^2=x$

$x(x-1)=0$, $x=0$, $x=1$, $O(0,0)$,
$y=0$, $y=0$, $A(1,1)$

to find new limits -

$y = x$ to $y = \sqrt{x}$

$x = 0$ to $x = 1$

$$I = \int_0^1 \frac{1}{1-x}\left[\int_x^{\sqrt{x}} \frac{y\,dy}{\sqrt{x-y^2}}\right] dx = \int_0^1 \frac{1}{1-x}(I_1)\,dx \quad \text{--- (2)}$$

where $I_1 = \int_x^{\sqrt{x}} \frac{y\,dy}{\sqrt{x-y^2}} = \frac{-1}{2}\int_x^{\sqrt{x}} \frac{-2y\,dy}{\sqrt{x-y^2}}$

$$= -\tfrac{1}{2}\left[2\sqrt{x-y^2}\right]_x^{\sqrt{x}}$$

$I_1 = \sqrt{x}\sqrt{1-x}$ (2)

from eqn (2)

$$I = \int_0^1 \frac{\sqrt{x}\sqrt{1-x}}{1-x}\,dx = \int_0^1 x^{1/2}(1-x)^{-1/2}\,dx$$

put $x = t$, $dx = dt$

Limits: x 0 1

t 0 1

$$I = \int_0^1 1^{1/2} t^{1/2} 1^{-1/2}(1-t)^{-1/2} 1\,dt$$

$$= 1\int_0^1 t^{1/2}(1-t)^{-1/2}\,dt$$

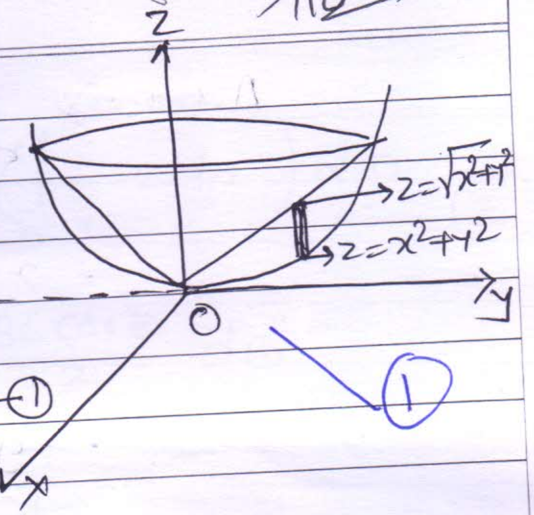
$$= \beta(3/2, 1/2) = \frac{\Gamma 3/2\,\Gamma 1/2}{2}$$

$$I = \frac{1/2\,\Gamma 1/2\,\Gamma 1/2}{1}$$

$$I = \frac{\pi}{2}$$ (2)

Q.4. b)

$$V = \iint \left(\int_{z=x^2+y^2}^{\sqrt{x^2+y^2}} dz \right) dx\,dy$$



1

$$V = \iint \left[\sqrt{x^2+y^2} - (x^2+y^2) \right] dx\,dy \longleftarrow (1)$$

changing eqn (1) into polar coordinates by putting

$x = r\cos\theta, \quad y = r\sin\theta$

$dx\,dy = r\,d\theta\,dr$

$$V = 4\int_0^{\pi/2}\int_0^1 (r - r^2)\, r\,d\theta\,dr$$ 1

$$= 4\int_0^{\pi/2} \left[\int_0^1 (r^2 - r^3)\,dr \right] d\theta$$

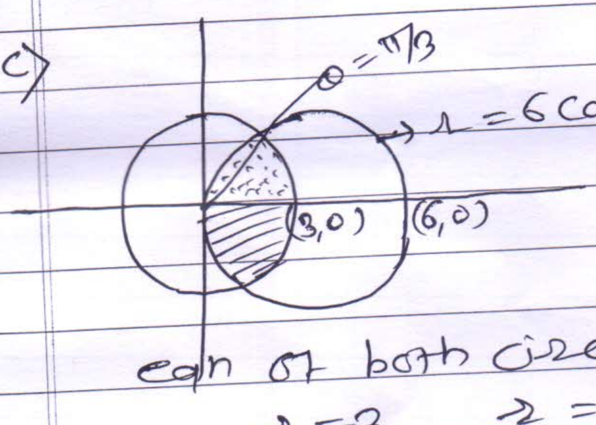
$$= 4\int_0^{\pi/2} \left[\frac{r^3}{3} - \frac{r^4}{4} \right]_0^1 d\theta$$

$$= 4\int_0^{\pi/2} \left(\frac{1}{3} - \frac{1}{4} \right) d\theta$$

$$= 4\int_0^{\pi/2} \frac{1}{12}\,d\theta = 4 \cdot \frac{1}{12} \cdot \frac{\pi}{2}$$

volume $= \pi/6$ —— 2

c)



Given circles are $x^2+y^2=9$, and $x^2+y^2-6x=0$

eqn of both circles in polar form is

$r = 3, \quad r = 6\cos\theta.$

$3 = 6\cos\theta, \quad \cos\theta = \frac{1}{2}, \quad \theta = \pi/3$

$$\text{Area} = 2\int_0^{\pi/3}\left(\int_0^3 r\,dr\right)d\theta + 2\int_{\pi/3}^{\pi/2}\left(\int_0^{6\cos\theta} r\,dr\right)d\theta \quad \text{— } 2$$

$$= 2\int_0^{\pi/3} 9/2\,d\theta + 2\int_{\pi/3}^{\pi/2} \frac{36\cos^2\theta}{2}\,d\theta$$

$$= 9\,\pi/3 + 18\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)\,d\theta$$

$$= 9\,\pi/3 + 18\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2}$$

$$= 9\,\pi/3 + 18\left[(\pi/2 - \pi/3) + \left(0 - 1/2\,\frac{\sqrt{3}}{2}\right)\right]$$

$$= 3\pi + 18\left[\pi/6 - \sqrt{3}/4\right]$$

$$= 3\pi + 3\pi - \frac{9\sqrt{3}}{2}$$

$$\text{Area} = 6\pi - \frac{9\sqrt{3}}{2} \quad \text{— } 2$$

Multiple choice questions : 20 marks

1) Ans: $\dfrac{y-xy_1}{y_1^2} = y_1$ where $y_1 = \dfrac{dy}{dx}$

$y - xy_1 = y_1^3$

order $=1$ and degree $=3$. —— 1 m

2) Ans: $y = ae^{2x} + be^{3x}$

$ye^{-3x} = ae^{-x} + b$

$e^{-3x}(-3y+y_1) = -ae^{-x}$

$e^{-2x}(-3y+y_1) = -a$

$e^{-2x}(6y-2y_1-3y_1+y_2) = 0 \Rightarrow e^{-2x} \neq 0$

$\therefore 6y - 5y_1 + y_2 = 0$, Hence $\boxed{k=5}$ —— 2 m

3) $\dfrac{dy}{dx}(\tan^{-1}y - x) = 1+y^2$

$\dfrac{dx}{dy}(1+y^2) = \tan^{-1}y - x$

$\dfrac{dx}{dy}(1+y^2) + x = \tan^{-1}y$

$\dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{\tan^{-1}y}{1+y^2}$

$P = \dfrac{1}{1+y^2} \therefore$ I.F. $= e^{\int P\,dy} = e^{\tan^{-1}y}$ —— 2 m

4) $x^2 + cy^2 = 1 \Rightarrow c = \dfrac{1-x^2}{y^2}$

$2x + c\cdot 2y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{-x}{cy}$, by substituting value of c

$\dfrac{dy}{dx} = \dfrac{-xy}{1-x^2} = \dfrac{xy}{x^2-1}$. —— 2 m

5) $t=0, \quad x_0 = 1.5$

$t = ?, \quad x = 0.5$

$x = 1.5e^{-kt}$ —— 1

$0.5 = 1.5e^{-kt}, \; e^{-kt} = 1/3$ —— 1 —— 2m

6) $L\frac{dI}{dt} + RI = E(t)$ ——— 1m.

7) $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ ——— 1m

8) $I = \int_0^a \sqrt{a^2-x^2}\,dx$

put $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$.

$= \int_0^{\pi/2} \sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$. 1

$= \int_0^{\pi/2} a^2\cos^2\theta\,d\theta = a^2\cdot\frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi a^2}{4}$ 1 ——— 2m

9) $\Gamma\frac{1}{3}\,\Gamma\left(1-\frac{1}{3}\right) = \Gamma\frac{1}{3}\,\Gamma\frac{2}{3} = \frac{\pi}{\sin p\pi} = \frac{\pi}{\sin\pi/3} = \frac{\pi}{\sqrt{3}/2}$ 1

$= \frac{2\pi}{\sqrt{3}}$ 1 ——— 2m

10) Lowest degree term $= (x-y)$ thus $x=y$ is the tangent at origin ——— 1m.

11) $r = \frac{2a}{1+\cos\theta}$.

$\tan\phi = \frac{a\sec^2\theta/2}{\frac{2a}{2}\sec\theta/2\,\sec\theta/2\,\tan\theta/2}$

$= \cot\theta/2 = \tan(\pi/2-\theta/2)$ ——— 2m

$\boxed{\phi = \pi/2 - \theta/2}$

12) $\frac{dx}{dy} = \frac{4y^3-4y^{-3}}{8} = \frac{1}{2}\left(y^3-\frac{1}{y^3}\right)$

Length of the curve

$s = \int_1^2 \sqrt{1+\left(\frac{1}{2}\left(y^3-\frac{1}{y^3}\right)\right)^2}\,dy$. 1

$= \int_1^2 \frac{y^6+1}{2y^3}\,dy = \frac{1}{2}\left[\frac{y^4}{4}-\frac{y^{-2}}{2}\right]_1^2 = \frac{33}{16}$ ——— 2m 1