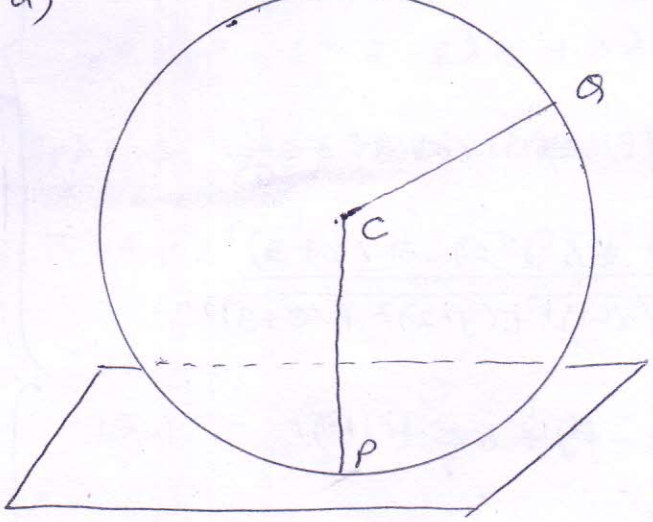
SOLUTIONS AND SCHEME OF MARKING

NOTE: GIVE FULL CREDIT TO ALTERNATIVE SOLUTIONS.

U118-109(BE-FF)

Q.1. a)



$P \equiv (1, 2, -2)$

$Q \equiv (-1, 0, 0)$

Let the required sphere be:

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

centre $C \equiv (-u, -v, -w)$

d.r.s of $CP \equiv 1+u,\ 2+v,\ -2+w$

d.r.s of normal to plane : $2, -1, -2$

CP $\perp$ to plane

$\therefore \frac{1+u}{2} = \frac{2+v}{-1} = \frac{-2+w}{-2} = k$

2 marks

$\therefore C \equiv (1-2k,\ 2+k,\ -2+2k)$

$l(CP) = \left| \frac{2(1-2k) - (2+k) - 2(-2+2k) - 4}{3} \right|$

$\therefore l(CP) = 3k$, $l(CP) = l(CQ) =$ radius

$3k = \sqrt{(2-2k)^2 + (2+k)^2 + (-2+2k)^2}$

$\Rightarrow \boxed{k=1}$ $\therefore$ centre $C \equiv (-1, 3, 0)$

2 marks

sphere eqn becomes : $x^2 + y^2 + z^2 + 2x - 6y + d = 0$

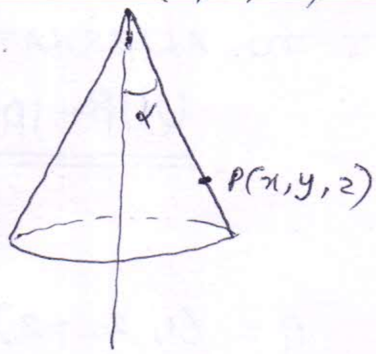
$Q(-1, 0, 0)$ lies on sphere $\therefore 1 - 2 + d = 0 \ \therefore \underline{d=1}$

2 marks

$\therefore$ Required sphere:

$\boxed{x^2 + y^2 + z^2 + 2x - 6y + 1 = 0}$

b)



d.r.s of generator VP $\equiv x-1,\ y-2,\ z+3$

d.r.s of Axis $\equiv 1,\ 2,\ -1$

4 marks

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$$

$$\frac{1}{\sqrt{3}} = \frac{1(x-1)+2(y-2)-1(z+3)}{\sqrt{6}\cdot\sqrt{(x-1)^2+(y-2)^2+(z+3)^2}}$$

$$2(x^2+y^2+z^2-2x-4y+6z+14)$$
$$= (x+2y-z-8)^2$$

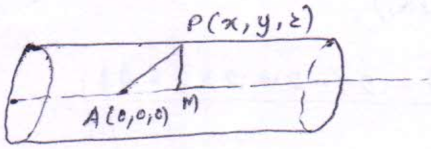
$$2x^2+2y^2+2z^2-4x-8y+12z+28$$
$$= x^2+4y^2+z^2+64+4xy-4yz-2xz$$
$$-16x-32y+16z.$$

2 marks.

Required cone:

$$\boxed{x^2-2y^2+z^2-4xy+4yz+2xz+12x+24y+4z-36=0}$$

c)



$PM = 4$, d.r.s of Axis $\equiv 2, 1, -2$

$AP = \sqrt{x^2+y^2+z^2}$

2 marks

$$AM = \frac{2}{3}(x-0)+\frac{1}{3}(y-0)+\left(-\frac{2}{3}\right)(z-0)$$

$$AM = \frac{2x+y-2z}{3}$$

$$(AM)^2+(PM)^2 = (AP)^2$$

$$\left(\frac{2x+y-2z}{3}\right)^2+4^2 = \left(\sqrt{x^2+y^2+z^2}\right)^2$$

Required cylinder:

2 marks.

$$\boxed{5x^2+8y^2+5z^2-4xy+4yz+8xz-144=0}$$

2. a) Equation of the sphere through given circle :

$$(x^2+y^2+z^2-9) + \lambda(2x+3y+4z-5) = 0 \quad \text{--- (1)} \qquad \text{— 2 marks}$$

$$\therefore \; x^2+y^2+z^2+2\lambda x+3\lambda y+4\lambda z-5\lambda-9=0$$

Sphere passes through the point $(1,2,3)$

$$\therefore \; 1^2+2^2+3^2+2\lambda(1)+3\lambda(2)+4\lambda(3)-5\lambda-9=0$$

$$15\lambda = -5 \Rightarrow \lambda = -\frac{1}{3}$$

2 marks

putting $\lambda = -\frac{1}{3}$ in Eq^n (1),

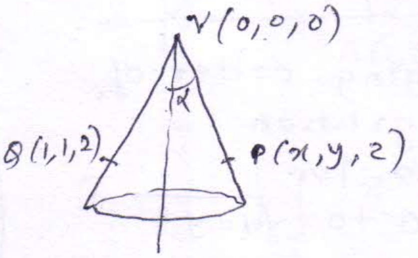
$$(x^2+y^2+z^2-9) - \frac{1}{3}(2x+3y+4z-5) = 0$$

$\therefore$ Required sphere :

$$\boxed{3(x^2+y^2+z^2) - 2x - 3y - 4z - 22 = 0}$$

2 marks

b)



d.r.s of Axis $\equiv 2, -4, 3$

d.r.s of VQ $\equiv 1, 1, 2$

$$\cos\alpha = \frac{(2)(1)-4(1)+3(2)}{\sqrt{29}\cdot\sqrt{6}}$$

$$\therefore \cos\alpha = \frac{4}{\sqrt{29}\cdot\sqrt{6}}$$

3 marks

d.r.s of VP $\equiv x, y, z$

$$\cos\alpha = \frac{(2)(x)-4(y)+3(z)}{\sqrt{29}\cdot\sqrt{x^2+y^2+z^2}}$$

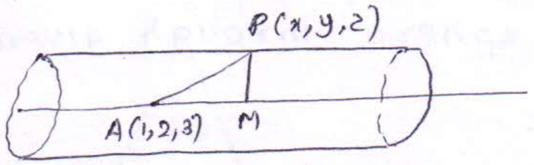
$$\frac{4}{\sqrt{29}\cdot\sqrt{6}} = \frac{2x-4y+3z}{\sqrt{29}\cdot\sqrt{x^2+y^2+z^2}}$$

Required cone :

$$\boxed{4x^2+40y^2+19z^2-48xy-72yz+36xz=0}$$

3 marks.

c)



$PM = 2$

$AP = \sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$

$$AM = \frac{2}{7}(x-1) - \frac{3}{7}(y-2) + \frac{6}{7}(z-3)$$

2 marks

$$AM = \frac{2x-3y+6z-14}{7}$$

$$(AM)^2 + (PM)^2 = (AP)^2$$

$$\left(\frac{2x-3y+6z-14}{7}\right)^2 + 2^2 = (x-1)^2+(y-2)^2+(z-3)^2$$

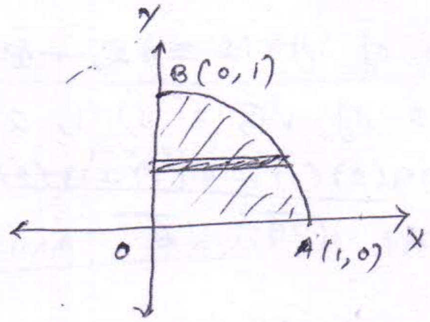
2 marks

$$4x^2+9y^2+36z^2+196-56x+84y-168z+196 - 12xy - 36yz + 24xz$$
$$= 49(x^2+y^2+z^2-2x-4y-6z+14)$$

Required cylinder :

$$\boxed{45x^2+40y^2+13z^2+12xy+36yz-24xz -42x-280y-126z+294=0}$$

Q.3 a)



changing order of integration:
Limits for
x : 0 to $\sqrt{1-y^2}$
y : 0 to 1

2 marks

$$I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} \frac{dx\,dy}{(1+e^y)\cdot\sqrt{1-x^2-y^2}}$$

$$= \int_{y=0}^{1}\left[\left(\frac{1}{1+e^y}\right)\int_{x=0}^{\sqrt{1-y^2}} \frac{dx}{\sqrt{1-y^2-x^2}}\right] dy$$

$$= \int_{y=0}^{1} \left(\frac{1}{1+e^y}\right)\left[\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right]_{x=0}^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \left(\frac{1}{1+e^y}\right)\left(\frac{\pi}{2}-0\right) dy$$

2 marks

$$= \frac{\pi}{2}\int_0^1 \frac{dy}{1+e^y} = \frac{\pi}{2}\int_0^1 \left(\frac{-e^{-y}}{-e^{-y}-1}\right) dy$$

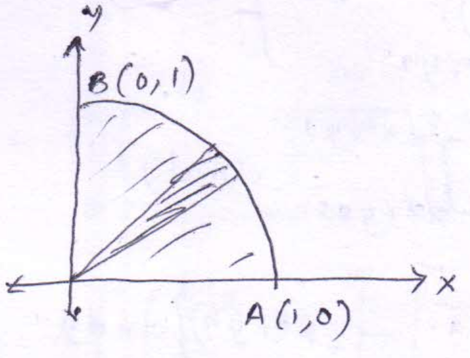
$$= \left(-\frac{\pi}{2}\right)\int_0^1 \left(\frac{-e^{-y}}{e^{-y}+1}\right) dy = \left(-\frac{\pi}{2}\right)\left[\log(e^{-y}+1)\right]_0^1$$

$$= \left(-\frac{\pi}{2}\right)\left[\log(e^{-1}+1) - \log 2\right]$$

$$\boxed{I = \frac{\pi}{2}\log\left(\frac{2e}{1+e}\right)}$$

2 marks

b)



Let $x = r\cos\theta$
$y = r\sin\theta$

$\therefore\ dx\,dy = r\,dr\,d\theta$

Limits: r : 0 to 1

θ : 0 to $\pi/2$

$$I = \iint_R x^2 y^2\, dx\,dy$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} (r\cos\theta)^2 (r\sin\theta)^2\, r\,dr\,d\theta$$

2 marks

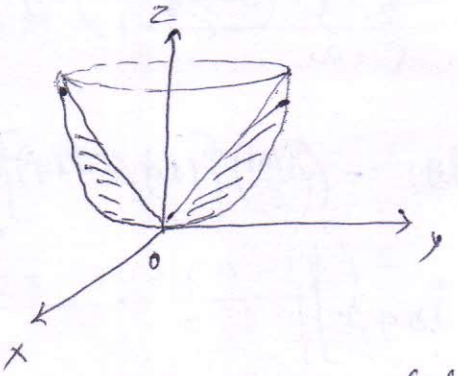
$$= \left[\int_{\theta=0}^{\pi/2} (\cos^2\theta \cdot \sin^2\theta)\, d\theta\right]\left[\int_{r=0}^{1} r^5\, dr\right]$$

$$\left[\frac{\quad}{(4)(2)}\quad{}^{2}\right]\left[\frac{\quad}{6}\right]_0$$

2 m

$$\boxed{I = \frac{\pi}{96}}$$

c)



Limits;

$z : x^2+y^2 \text{ to } \sqrt{x^2+y^2}$

Let, $V = \iiint dx\,dy\,dz$

$$= \iint \left[\int_{x^2+y^2}^{\sqrt{x^2+y^2}} dz\right] dx\,dy$$

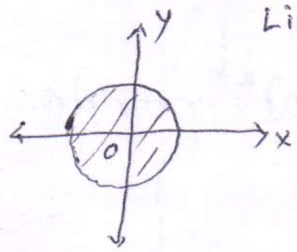
$$= \iint \left[z\right]_{x^2+y^2}^{\sqrt{x^2+y^2}} dx\,dy$$

$$V = \iint \left[\sqrt{x^2+y^2} - (x^2+y^2)\right] dx\,dy$$

2 marks

Let $x = r\cos\theta,\ y = r\sin\theta$

$\therefore dx\,dy = r\,dr\,d\theta$



Limits: $r : 0 \text{ to } 1$

$\theta : \underbrace{0 \text{ to } \pi/2}_{4 \text{ times}}$

$$V = 4\int_{r=0}^{1}\int_{0}^{\pi/2} (r - r^2)\, r\,dr\,d\theta$$

$$= 4\left[\int_{r=0}^{1}(r^2 - r^3)\,dr\right]\left[\int_{0}^{\pi/2} d\theta\right]$$

(7)

$$= 4\left[\frac{r^3}{3} - \frac{r^4}{4}\right]_0^1 \left[\theta\right]_0^{\pi/2}$$

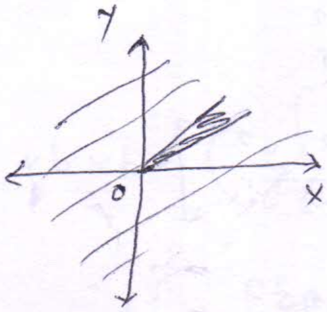
$$= 4\left[\frac{1}{3} - \frac{1}{4}\right]\left[\frac{\pi}{2}\right]$$

$$\therefore \boxed{V = \frac{\pi}{6}}$$

2 marks

Q.4 a)

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{(1+x^2+y^2)^{3/2}}$$



Let $x = r\cos\theta$
$y = r\sin\theta$

$\therefore dx\,dy = r\,dr\,d\theta$

Limits : r : 0 to ∞

θ : $\underbrace{0 \text{ to } \pi/2}_{4 \text{ times}}$

3 marks

$$I = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} \frac{r\,dr\,d\theta}{(1+r^2)^{3/2}}$$

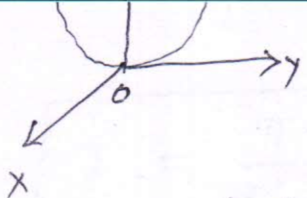
$$= 4\left[\int_{\theta=0}^{\pi/2} d\theta\right]\left[\int_{r=0}^{\infty} \frac{r\,dr}{(1+r^2)^{3/2}}\right]$$

$$= 4\left[\theta\right]_0^{\pi/2}\left[-\frac{1}{\sqrt{1+r^2}}\right]_0^{\infty}$$

$$= 4\left(\frac{\pi}{2}\right)(1)$$

$$\boxed{I = 2\pi}$$

3 marks



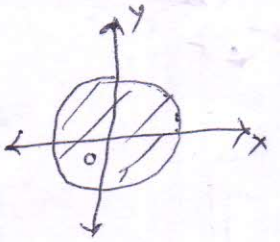
$$I = \iint \left[\int_{z = \frac{x^2+y^2}{2}}^{2} (x^2+y^2)\, dz \right] dx\, dy$$

$$= \iint (x^2+y^2) \Big[z \Big]_{\frac{x^2+y^2}{2}}^{2} dx\, dy$$

$$= \iint (x^2+y^2) \left[2 - \left(\frac{x^2+y^2}{2} \right) \right] dx\, dy$$

$$I = \iint \left[2(x^2+y^2) - \frac{(x^2+y^2)^2}{2} \right] dx\, dy$$

2 marks



Let $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

Limits : r : 0 to 2

θ : $\underbrace{0 \text{ to } \pi/2}_{4 \text{ times}}$

$$I = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} \left[2r^2 - \frac{r^4}{2} \right] r\, dr\, d\theta$$

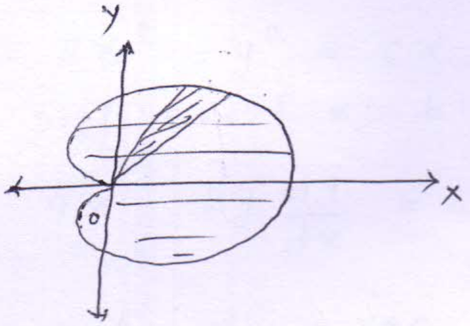
$$= 4 \left[\int_{\theta=0}^{\pi/2} d\theta \right] \left[\int_{r=0}^{2} \left(2r^3 - \frac{r^5}{2} \right) dr \right]$$

$$= 4 \Big[\theta \Big]_0^{\pi/2} \left[\frac{2r^4}{4} - \frac{r^6}{12} \right]_0^2$$

$$= 4 \left(\frac{\pi}{2} \right) \left(\frac{32}{4} - \frac{64}{12} \right)$$

$$\boxed{I = \frac{16\pi}{3}}$$

2 marks



Area, $A = \iint r\,dr\,d\theta$

Limits: r : 0 to $a(1+\cos\theta)$

θ : $\underbrace{0 \text{ to } \pi}_{\text{twice}}$

2 marks

$$A = 2\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)}$$

$$= a^2\int_0^{\pi}(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi}\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta$$

$$= 4a^2\int_0^{\pi/2}(\cos^4 t)\,2\,dt$$

put $\frac{\theta}{2} = t$

$d\theta = 2dt$

$$= 8a^2\left[\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}\right]$$

$$\boxed{A = \frac{3\pi a^2}{2}}$$

2 marks

Q.5 a) $y = c^2 + \frac{c}{x}$

$$\frac{dy}{dx} = -\frac{c}{x^2}$$

$$\Rightarrow c = -x^2 \frac{dy}{dx}$$

1 mark

$$\therefore y = \left(-x^2 \frac{dy}{dx}\right)^2 + \frac{1}{x}\left(-x^2 \frac{dy}{dx}\right)$$

$$\boxed{y = x^4 \left(\frac{dy}{dx}\right)^2 - x \frac{dy}{dx}}$$

1 mark

b) $y = A\cos x + B\sin x$

$$\frac{dy}{dx} = -A\sin x + B\cos x$$

1 mark

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x$$

$$\frac{d^2y}{dx^2} = -y$$

$$\boxed{\frac{d^2y}{dx^2} + y = 0}$$

1 mark

c) $\frac{dy}{dx} + \frac{y}{x^2} = x^2$

$$P = \frac{1}{x^2}$$

$$I.F = e^{\int P\,dx} = e^{\int \frac{1}{x^2}\,dx}$$

1 mark

$$-\frac{1}{x}$$

1 mark

$$4x - 2y\frac{dy}{dx} = C$$ 1 mark

$\therefore$ D.E of given curves:

$$2x^2 - y^2 = \left(4x - 2y\frac{dy}{dx}\right)x$$

for DE of orthogonal trajectories:

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$\therefore 2x^2 - y^2 = \left[4x - 2y\left(-\frac{dx}{dy}\right)\right]x$$ 1 mark

$$\therefore 2x^2 - y^2 = 4x^2 + 2xy\frac{dx}{dy}$$

$$\therefore \boxed{2xy\frac{dx}{dy} + 2x^2 + y^2 = 0}$$

e) $$\frac{d\theta}{\theta - 20} = -k\,dt$$

$\Rightarrow \log(\theta - 20) = -kt + C$ 1 mark

$\log 80 = C$ $\quad\therefore \log(\theta - 20) = -kt + \log 80$

At $t = 15$, $\theta = 80^\circ C$

$\log(80 - 20) = -15k + \log 80$ 1 mark

$$\therefore k = -\frac{1}{15}\log\left(\frac{6}{8}\right) = \underline{\frac{1}{15}\log\left(\frac{4}{3}\right)}$$

f) $$I = \int_0^\infty e^{-4x}x^4\,dx$$ 1 mark

we know, $\int_0^\infty e^{-kx}x^{n-1}\,dx = \frac{\sqrt{n}}{k^n}$

$$\therefore I = \frac{\sqrt{5}}{4^5} = \underline{\frac{15}{128}}$$ 1 mark

$$= \sqrt{(0.5)^2 + (1.5)^2}$$

$$= 1.58$$

} 1 mark

h) $$I(a) = \int_0^{a^2} \sin^{-1}\left(\frac{x}{a}\right) dx$$

$$\frac{dI}{da} = \int_0^{a^2} \left[\frac{\partial}{\partial a} \sin^{-1}\left(\frac{x}{a}\right)\right] dx + \frac{d}{da}(a^2) \cdot \sin^{-1}\left(\frac{a^2}{a}\right) - 0$$

— 1 mark

$$\frac{dI}{da} = \int_0^{a^2} \frac{1}{\sqrt{1-(x/a)^2}} \left(-\frac{x}{a^2}\right) dx + 2a \cdot \sin^{-1}(a)$$

$$= \frac{1}{a} \int_0^{a^2} \frac{-x}{\sqrt{a^2 - x^2}} dx + 2a\sin^{-1}a$$

$$= \frac{1}{a}\left[\sqrt{a^2 - x^2}\right]_0^{a^2} + 2a\sin^{-1}a$$

$$= \frac{1}{a}\left[\sqrt{a^2 - a^4} - a\right] + 2a\sin^{-1}a$$

$$\frac{dI}{da} = \sqrt{1 - a^2} - 1 + 2a\sin^{-1}a$$

} 1 mark

i) $$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$ — 1 mark

$$\frac{d}{dx}\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\left[\int_0^x \left[\frac{\partial}{\partial x} e^{-u^2}\right] du + \frac{d}{dx}(x) \cdot e^{-x^2} - 0\right]$$

$$\frac{d}{dx}\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$ — 1 mark

1) 1) Option a : Two branches have distinct tangents ——— 1 mark

2) Option c : $f(t)$ is an odd function and $g(t)$ is even function of t — 1 mark