Q1) a) $(D^2+4)\,y = \tan(2x)$.

$y_c = c_1\cos(2x) + c_2\sin(2x)$ ——— (1M)

$y_1 = \cos(2x)$, $y_2 = \sin(2x)$, $W = 2$. ——— (1M)

$\therefore u = \int \frac{-y_2 X}{W}\,dx = \frac{1}{4}\left[\sin(2x) - \log(\sec(2x)+\tan(2x))\right]$ ——— (2M)

$v = \int \frac{y_1 X}{W}\,dx = \frac{-1}{4}\cos(2x)$ ——— (2M)

$y_p = uy_1 + vy_2 = \frac{-1}{4}\cos(2x)\cdot\log\left[\sec(2x)+\tan(2x)\right]$

b) Put $x = e^z$, $z = \log x$.

$\therefore (D(D-1) - 3D + 5)y = e^{2z}\cdot z$ ——— (1M)

$(D^2 - 4D + 5)y = z e^{2z}$, $D = \frac{+4 \pm \sqrt{16-20}}{2} = 2 \pm i$. ——— (1M)

$\therefore y_c = e^{2z}(c_1\cos z + c_2\sin z) = x^2(c\cos(\log x) + c_2\sin(\log x))$

$y_p = \frac{1}{D^2-4D+5}\, z e^{2z} = e^{2z}\frac{1}{(D+2)^2 - 4(D+2) + 5}\, z$

$= e^{2z}\left(\frac{1}{D^2+1}\, z\right) = e^{2z}\left[(1 - D^2 + D^4 \dots)\, z\right] = z e^{2z}$.

$\boxed{y_p = (\log x)\cdot x^2}$ ——— (4M)

$y = y_c + y_p$

Q2) a) $f(x) = \begin{cases} x^2 & 0<x<a \\ 0 & x>a \end{cases}$, F.C.T. $= \int_0^\infty f(u)\cdot\cos(\lambda u)\,du = F_c(\lambda)$ ——— (2M)

$F_c(\lambda) = \int_0^a u^2\cos(\lambda u)\,du = \left[u^2\frac{\sin(\lambda u)}{\lambda} + 2u\frac{\cos(\lambda u)}{\lambda^2} + 2\frac{\sin(\lambda u)}{\lambda^3}\right]_0^a$

$= \frac{a^2\sin(a\lambda)}{\lambda} + \frac{2a\cos(a\lambda)}{\lambda^2} + \frac{2\sin(a\lambda)}{\lambda^3} - \frac{2a}{\lambda^2}$. ——— (4M)

(b) $6f(k+2) - 5f(k+1) + f(k) = 0$, $k \geq 0$, $f(0) = 0$, $f(1) = 3$.

$6\left[z^2F(z) - z^2f(0) - zf(1)\right] - 5\left[zF(z) - zf(0)\right] + F(z) = 0$.

$(6z^2 - 5z + 1)F(z) - 3z = 0$. ——— (1M)

[redacted]

$f(k) = \frac{3}{5}(1)^k - \frac{1}{10}\left(\frac{1}{6}\right)^k, \quad k \geq 0.$ ——— 2M

Q3) a) First four moments

Values from Table, $h = 0.5$, $a = 3.5$.

$\Sigma f_i = 310$, $\Sigma f_i u_i = 36$, $\Sigma f_i u_i^2 = 560$.
$\Sigma f_i u_i^3 = 204$, $\Sigma f_i u_i^4 = 2480$. ——— 2M

$\mu_1' = 0.058065$ $\quad \mu_3' = 0.082259$
$\mu_2' = 0.451613$ $\quad \mu_4' = 0.5$ ——— 1M

$\mu_1 = 0$, $\mu_2 = 0.448242$, $\mu_3 = 0.00398$, $\mu_4 = 0.489997$ ——— 1M

$\beta_1 = 1.76 \times 10^{-4}$, $\beta_2 = 2.445$. ——— 1M

b)

$\Sigma X = 56$, $\Sigma Y = 40$.

$\Sigma XY = 364$, $\Sigma X^2 = 524$, $\Sigma Y^2 = 256$. ——— 2M

$\bar{X} = 7$, $\bar{Y} = 5$. $\quad r = 0.9770$ ——— 2M

$Cov = 84$, $\sigma_x = \sqrt{132}$, $\sigma_y = \sqrt{56}$.

$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$

& $(x - \bar{x}) = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$. ——— 2M

Q4) a) $\nabla \times \bar{F} = \bar{0}$. ——— 2M

$\phi = y^2 \sin x + x z^2 + c$ ——— 2M

b) $\phi = e^{2x-y-z}$, $\nabla\phi = 2e^{2x-y-z}\bar{i} - e^{2x-y-z}\bar{j} - e^{2x-y-z}\bar{k}$.

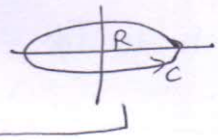
$(\nabla\phi)_{(1,1,1)} = 2\bar{i} - \bar{j} - \bar{k}$. ——— 1M

$\bar{r} = e^t\cos t\,\bar{i} + e^t\sin t\,\bar{j} + e^t\bar{k}$, $\quad \frac{d\bar{r}}{dt} = e^t(\cos t - \sin t)\bar{i} + e^t(\sin t + \cos t)\bar{j} + e^t\bar{k}$.

$\left(\frac{da}{dt}\right)_{t=0} = i + j + k.$ $\quad a$ ——— (1M)

$\therefore$ dd. $= \nabla\phi \cdot \hat{a} = \frac{2-1-1}{\sqrt{3}} = 0.$ ——— (2M)

Q5) a)



ellipse, $z = 0$. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$, $\bar{F} = \sin y\, \hat{i} + x(1+\cos y)\hat{j}$

(2M)

Green's $Th^m = \int_C \bar{F} \cdot d\bar{r} = \iint_R \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\, dx\, dy$ ——— (2M)

$= \iint 1 + \cos y - \cos y \; dx\, dy$

$= \iint dx\, dy = \pi ab$ ——— (2M)

(b) Work done $= \int \bar{F} \cdot d\bar{r}$

$c = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0,\ dz = 0$

$= \int_C (2x - y)\, dx + (x + y)\, dy$ ——— (2M)

Put $x = 5\cos\theta,\ y = 4\sin\theta$

$dx = -5\sin\theta\, d\theta, \quad dy = 4\cos\theta\, d\theta.$

θ varies from 0 to 2π.

$W = \int_0^{2\pi} (10\cos\theta - 4\sin\theta)(-5\sin\theta)\, d\theta + (5\cos\theta + 4\sin\theta)\, 4\cos\theta\, d\theta$

$= 40\pi.$ ——— (2M)

(c) By divergence Th^m

$\nabla \cdot \frac{\bar{r}}{r^2} = \frac{(\nabla \cdot \bar{r})}{r^2} + (\nabla r^{-2}) \cdot \bar{r} = \frac{3}{r^2} + \frac{(-2)}{r^3}(\bar{r}, \bar{r}) = \frac{1}{r^2}$ ——— (2M)

$\therefore \iiint_V \frac{dv}{r^2} = \iint_S \frac{\bar{r}}{r^2} \cdot \hat{n}\, ds$

——— (2M)

[redacted]

$\nabla \cdot \bar{F} = 3(x^2+y^2+z^2)$. —— (2M)

Put $x = \cancel{a} r\sin\theta \cdot \cos\phi$

$y = \cancel{b} r\sin\theta \cdot \sin\phi$ $\quad dx\,dy\,dz = r^{\cancel{3}}\sin\theta\, dr\, d\theta\, d\phi$

$z = r\cos\theta$. —— (2M)

$$\therefore \text{R.H.S.} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{0}^{4} 3r^2 \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= (\phi)_0^{2\pi} \; (-\cos\theta)_0^{\pi} \; \left(\frac{3r^5}{5}\right)_0^4$$

$$= -2\pi \cdot [-1-\cancel{1}] \; \frac{3}{5}(16\times16) = \frac{12\pi}{5}(4^5).$$ —— (2M)

(b) $\bar{F} = (2x+y^2)\bar{i} + (3y-4x)\bar{j}$

$$\int_{y=x^2} F\cdot d\bar{r} = \int_0^1 (2x+x^4)\,dx + (3x^2-4x)\cdot 2x\,dx$$ —— (2M)

x varies from 0 to 1

$$= \left(-2\frac{x^2}{2} + \frac{x^5}{5} + \frac{3x^3}{3}\right)_0^1 = -1+\frac{1}{5}+1 = \frac{1}{5}$$ —— (2M)

(c) Stoke's th^m.

S: $x^2+4y^2+z^2-2x = \cancel{4} \to$ ellipsoid. cut by plane $x=0$.

give $\Rightarrow$ $4y^2+z^2=4$. $\Rightarrow \frac{y^2}{1}+\frac{z^2}{4}=1$ $\Rightarrow$ ~~ell~~ ~~semi~~ ellipse.

i.e. open surface is bdd by above ellipse.

$$\therefore \int_C F\cdot d\bar{r} = \iint_S (\nabla\times F)\cdot\hat{n}\,ds$$ —— (2M)

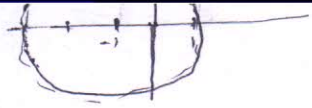
$$\therefore \text{L.H.S.} = \int_{\text{ellipse},\, x=0,\, dx=0} f_1\,dx + f_2\,dy + f_3\,dz = \int y^3\,dz$$

$\frac{y^2}{1}+\frac{z^2}{4}=1$.

Put $y=\cos\theta$, $z=2\sin\theta$, $dz = 2\cos\theta\,d\theta$.

$$\text{L.H.S.} = \int_0^{2\pi}\cos^3\theta\cdot 2\cos\theta\,d\theta = \int_0^{2\pi} 2\cos^4\theta\,d\theta = \cancel{\frac{3}{4}}\times$$

$$= 8\int_0^{\pi/2}\cos^4\theta\,d\theta = 8\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{3\pi}{2}.$$ —— (2M)



$$f(z) = \frac{2z^2+2z+1}{(z+1)^2(z-3)}$$

$z=-1$ is multiple pole lie inside circle of order 2 —— (2M)

$z=3$ is pole does not lie inside circle.

$\therefore$ Res at $z=-1 \Rightarrow \frac{1}{1!}\left[\frac{d}{dz}(z+1)^2 \cdot f(z)\right]_{z=-1} = \frac{d}{dz}\left(\frac{2z^2+2z+1}{z-3}\right)$ —— (1M)

$$= \left[\frac{(z-3)(2z+2) - (2z^2+2z+1)}{(z-3)^2}\right]_{z=-1}$$

$$= \frac{(-4)(0) - 1}{(-4)^2} = \frac{-1}{16}$$

—— (2M)

$\therefore \int_C f(z)\,dz = 2\pi i\,(\text{Res at } z=-1) = \frac{-\pi i}{8}$

(b) $u - v = (x-y)(x^2+4xy+y^2)$

$u_x - v_x = (x-y)(2x+4y) + (x^2+4xy+y^2)$ —— (I)

C-R eq^{n}s are $u_x = v_y$ & $v_x = -u_y$

$u_y - v_y = (x-y)(4x+2y) - (x^2+4xy+y^2)$ —— (II)

$-v_x - u_x =$ —— " —— (III)

Add I & III.

$\therefore -2v_x = (x-y)[6x+6y] = 6(x^2-y^2)$ —— (1M)

$-v_x = 3(x^2-y^2)$, $\boxed{v_x = 3(y^2-x^2)}$

Subtract III from I.

$\therefore 2u_x = (x-y)(-2x+2y) + 2(x^2+4xy+y^2)$

$= -2x^2+2xy+2xy-2y^2+2x^2+8xy+2y^2$

$2u_x = 12xy \Rightarrow \boxed{u_x = 6xy}$ —— (1M)

$f'(z) = u_x + i v_x = 6xy + i\,3(y^2-x^2)$

Put $x=z$, $y=0$

$f'(z) = 3z^2 \Rightarrow \boxed{f(z) = z^3 + c}$ —— (2M)

$\therefore\ w = \dfrac{az+b}{cz+d}$, $i = \dfrac{b}{d} \Rightarrow id = b$.

(2M)

$-1 = \dfrac{a+b}{c+d} \Rightarrow -c-d = a+b \Rightarrow -c-d = a+id$.

$w = \dfrac{\frac{a}{b}+\frac{1}{z}}{\frac{c}{b}+\frac{d}{b}\frac{1}{z}} \Rightarrow -i = \dfrac{\frac{a}{b}}{c/b} \Rightarrow -ci = a. \Rightarrow -c-d = -ci+id$

Ans. (2M)

Q8) a) $f(z) = u+iv$, $|f(z)|^2 = u^2+v^2$.

$\dfrac{\partial}{\partial x}|f(z)|^2 = 2uu_x + 2vv_x$

$\dfrac{\partial^2}{\partial x^2} = 2uu_{xx} + 2u_x^2 + 2v_{xx}\cdot v + 2v_x^2$ — I (1M)

lly $\dfrac{\partial^2}{\partial y^2}|f'(z)|^2 = 2uu_{yy} + 2u_y^2 + 2vv_{yy} + 2v_y^2$ — II (1M)

Add I & II.

$\therefore$ L.H.S. $= 2u\underbrace{(u_{xx}+u_{yy})}_{0} + 2v\underbrace{(v_{xx}+v_{yy})}_{0} + 4(u_x^2+v_x^2+u_y^2+v_y^2)$

$+ 4|f'(z)|^2$ (2M)

bcoz $f'(z) = u_x + iv_x = v_y - iu_y$. (1M)

$\therefore |f'(z)|^2 = u_x^2+v_x^2 = u_y^2+v_y^2$ (1M)

(b) $\displaystyle\int_{\substack{C\\|z+2|=2}} \frac{e^z}{z+2}\,dz = 2\pi i\, f(-2) \Rightarrow 2\pi i\, e^{-2}$ (2M) $\because -2$ lie inside circle $|z+2|=2$. (2M)

(c) $u = 3x^2-3y^2+2y$, $u_x = 6x$, $u_y = -6y+2$.

$\therefore dv = v_x dx + v_y dy = -(2-6y)dx + 6x\,dy$

$v = -2x+6xy+c$ (2M)

$\therefore f(z) = u+iv = (3x^2-3y^2+2y) + i(6xy-2x)$.

Put $x=z$, $y=0$

$f(z) = 3z^2 - 2zi$. (2M)