ENGG MATHS - I ES11187

SOLUTIONS AND SCHEME OF MARKING ①

NOTE: FULL CREDIT TO BE GIVEN FOR ALTERNATIVE SOLUTIONS.

Paper code - UM18-101 (T1

Q.1 a) $A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & 1 & -3 & 4 \end{bmatrix}$

$R_3 + R_1$

$\sim \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ 0 & 2 & 0 & 10 \end{bmatrix}$

$(1/8)\, C_1$

$\sim \begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ 0 & 2 & 0 & 10 \end{bmatrix}$

$C_2 - C_1$; $C_3 - 3C_1$; $C_4 - 6C_1$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 2 \\ 0 & 2 & 0 & 10 \end{bmatrix}$

$(1/2)\, C_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 2 \\ 0 & 2 & 0 & 10 \end{bmatrix}$

$C_2 - 3C_3$; $C_4 - 2C_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 10 \end{bmatrix}$

$C_4 - 5C_2$ & then $(\frac{1}{2})C_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

3 marks

C_{23}

$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $\therefore A \sim [I_3 : 0]$, $\rho(A) = 3$ 1 mark

b) In matrix form:

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \quad X \quad = Z$$

For system to have non-trivial solutions

$$|A| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 6)(\lambda^2 + 3\lambda + 3) = 0$$

$$\therefore \boxed{\lambda = 6}$$

2 marks

for $\lambda = 6$

$$A = \begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}$$

solving first two equations by cramer's rule

$$\frac{x_1}{\begin{vmatrix} 2 & 3 \\ -5 & 2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 3 & -5 \\ 2 & 3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -5 & 2 \\ 3 & -5 \end{vmatrix}} = t \neq 0$$

2 marks

$$\therefore \quad x_1 = 19t \; ; \quad x_2 = 19t \; ; \quad x_3 = 19t.$$

$$\therefore \boxed{\begin{array}{l} x_1 = t, \quad x_2 = t, \quad x_3 = t \\ \text{where } t \neq 0 \end{array}}$$

3

c). $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

I) Characteristic Equation:

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$S_1 = -9 + 3 + 7 = 1$

$S_2 = -11 + 1 + 5 = -5$

$|A| = 3$

$\therefore \lambda^3 - \lambda^2 - 5\lambda - 3 = 0$

II) Eigen values:

$\lambda_1 = 3 \; ; \; \lambda_2 = -1 = \lambda_3$

2 marks

III) Eigen vector for $\lambda = 3$.

$\therefore (A - \lambda I)X = 0$

$\rightarrow \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

solving first two equations by Cramer's rule

$$\frac{x_1}{\begin{vmatrix} 4 & 4 \\ 0 & 4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 4 & -12 \\ 4 & -8 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -12 & 4 \\ -8 & 0 \end{vmatrix}} = t \neq 0$$

$\therefore x_1 = 16t \; ; \; x_2 = 16t \; ; \; x_3 = 32t$

$\therefore X = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

2 mark

Q. 2 a). To prove: $\left(\frac{101}{100}\right)^{100} < e < \left(\frac{100}{99}\right)^{100}$

Let $f(x) = e^x$ in $[a, b]$

i) $f(x)$ is continuous in $[a,b]$

ii) $f(x)$ is differentiable in (a, b), $f'(x) = e^x$ 2 mark

$\therefore$ By LMVT $\exists$ $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b-a} = f'(c)$$

$$\therefore \quad \frac{e^b - e^a}{b-a} = e^c$$

Now, $a < c < b$

$$e^a < e^c < e^b$$

$$e^a < \left(\frac{e^b - e^a}{b-a}\right) < e^b$$

firstly

$$e^a < \left(\frac{e^b - e^a}{b-a}\right)$$

gives $(b - a + 1) < e^{b-a}$

secondly

$$\left(\frac{e^b - e^a}{b-a}\right) < e^b$$

gives $e^{b-a} < \frac{1}{1-(b-a)}$ 2 marks

$$\therefore \quad (b-a+1) < e^{b-a} < \frac{1}{1-(b-a)}$$

putting $b-a = \frac{1}{100}$

$$\frac{101}{100} < e^{\frac{1}{100}} < \frac{100}{99}$$

$$\therefore \quad \left(\frac{101}{100}\right)^{100} < e < \left(\frac{100}{99}\right)^{100}$$

b) $f(x) = 3x^3 - 2x^2 + x - 4$

Taylor's expansion in ascending powers of $(x-a)$:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots \quad \text{——} (I)$$

$f(a) = 3a^3 - 2a^2 + a - 4$

$f'(a) = 9a^2 - 4a + 1$

$f''(a) = 18a - 4$

$f'''(a) = 18$

2 mark

Here, $x - a = x + 2 \Rightarrow \underline{a = -2}$

$$\left.\begin{array}{l} f(-2) = -38 \\ f'(-2) = 45 \\ f''(-2) = -40 \\ f'''(-2) = 18 \end{array}\right\} (II)$$

Putting (II) in (I)

$$3x^3 - 2x^2 + x - 4 = -38 + 45(x+2) - 40\frac{(x+2)^2}{2!} + 18\frac{(x+2)^3}{3!}$$

$\therefore$ $3x^3 - 2x^2 + x - 4$

$$\underline{= -38 + 45(x+2) - 20(x+2)^2 + 3(x+2)^3}$$

2 marks

c) $$\lim_{x \to 0} \frac{(a+b\cos x)x - c\sin x}{x^5} = 1$$

$$\lim_{x \to 0} \left[\frac{ax + b\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right)x - c\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)}{x^5} \right]$$

$= 1$

2 marks

$\Rightarrow \quad a + b - c = 0$ ——— (i)

$-3b + c = 0$ ——— (ii)

$5b - c = 120$ ——— (iii)

2 marks

Solving (i), (ii), (iii)

$\underline{a = 120}$; $\underline{b = 60}$; $\underline{c = 180}$

Q.3 a) Here, $a_n = \left[\frac{1 \cdot 2 \cdot 3 \cdots n}{4 \cdot 7 \cdots (3n+1)}\right] x^n$

$\therefore \quad a_{n+1} = \left[\frac{1 \cdot 2 \cdot 3 \cdots n \cdot (n+1)}{4 \cdot 7 \cdots (3n+1)(3n+4)}\right] x^{n+1}$

$\therefore \quad \frac{a_n}{a_{n+1}} = \left(\frac{3n+4}{n+1}\right) \cdot \frac{1}{x} = \left(\frac{3 + 4/n}{1 + 1/n}\right) \frac{1}{x}$

$\therefore \quad \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \frac{3}{x}$

By Ratio test
series converges for $\frac{3}{x} > 1 \Rightarrow x < 3$
& diverges for $\frac{3}{x} < 1 \Rightarrow x > 3$

For $x = 3$, ratio test fails.

2 marks

For $x = 3$.

$$\frac{a_n}{a_{n+1}} = \frac{3n+4}{3n+3}$$

$$\therefore \lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}} - 1\right)\right] = \lim_{n\to\infty}\left(\frac{n}{3n+3}\right) = \frac{1}{3} < 1.$$

$\therefore$ For $x = 3$ by Raabe's test given series diverges.

Hence given series is convergent for $0 < x < 3$ and divergent for $x \geq 3$.

2 mark

b) $f(x) = \pi^2 - x^2$

is an even function in the interval $-\pi \leq x \leq \pi$.

$\therefore$ $b_n = 0$.

Required Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

1 mark

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\,dx$$

$$= \frac{2}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\pi^3 - \frac{\pi^3}{3}\right]$$

$$a_0 = \frac{4\pi^2}{3}$$

1 mark

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi^2 - x^2) \cos nx \, dx$$

$$= \frac{2}{\pi} \left[(\pi^2 - x^2)\left(\frac{\sin nx}{n}\right) - (-2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\frac{2\pi}{n^2} \cdot \cos n\pi \right]$$

$$a_n = -\frac{4}{n^2} (-1)^n$$

2 marks

$\therefore$ Fourier series for given function is,

$$\pi^2 - x^2 = \frac{2\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$$